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Real Interpolation with Constraints*

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Given Banach spaces X, a subspace Y, and a finite set $\mathscr{G}$ of bounded linear functionals on Y, let $Y_{\mathscr{G}}$ denote all elements of Y which are anihilated by the functionals in $\mathscr{G}$. We investigate the relation between the real interpolation spaces $(X, Y)_{\theta,\rho}$ and $(X, Y_{\mathscr{G}})_{\theta,\rho}$. Applications are given to Sobolev spaces, best approximation with constraints, and weighted Lebesgue spaces. © 1995 Academic Press, Inc.

1. INTRODUCTION

This paper deals with the so called real interpolation method. We use [1] as a general reference to interpolation theory.

Thus let X and Y be two Banach spaces. Assuming that Y is continuously embedded in X, we consider the interpolation space $(X, Y)_{\theta,\rho}$. If $\mathscr{G} = \{\Gamma_1, ..., \Gamma_M\}$ is a given set of bounded linear functionals on Y, we let $Y_{\mathscr{G}}$ denote the subspace of Y of all elements u satisfying the constraints $\Gamma_1(u) = 0, ..., \Gamma_M(u) = 0$. In other words

$$Y_{\mathscr{G}} = Y \cap \left(\bigcap_{j=1}^{M} \ker(\Gamma_j) \right).$$

We shall investigate the relation between the two interpolation spaces $(X, Y)_{\theta,\rho}$ and $(X, Y_{\mathscr{G}})_{\theta,\rho}$. The later space is what we call an interpolation space with constraints. The relationship between the two spaces is described in an explicit form, which can be useful in several applications. Let us mention boundary value problems for partial differential equations, whereby the boundary operators are incorporated into the norm of the interpolation spaces. (See Löfström [4] and [5]. Cf. Grisvard [2], [3], Zolesio [8]). Another application is best approximation with constraints. (See Löfström [6]).

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In section 2 we give a general theorem on the extension of linear functionals which is necesary for the sequal. In section 3 we present a general theory giving the relation between the two interpolation spaces. Clearly some conditions on the functionals are needed. These conditions can be considered as a kind of independence, which we call strong independence. In section 4 we give sufficient conditions for strong independence. In the remaining two sections we discuss some general applications. In section 5 we consider interpolation of Sobolev spaces with applications to best approximation with constraints. In the last section we consider weighted Lebesque spaces.

The paper is an attempt to build a general theory which is able to cover various special cases considered in earlier papers. Here let us mention Löfström [4], [6]. Similar situations was considered by Grisvard [2], [3], Löfström [5], Thomee [7], Zolesio [8].

2. Extension of Linear Functionals

Let X and Y be two Banach spaces and assume that Y is continuously embedded in X. If Γ is a bounded linear functional on Y we introduce the functional

$$N(t, \Gamma) = \sup\{ |\Gamma(u)| : u \in Y, J(t, u) \leqslant 1\},$$

where $0 \leqslant t < \infty$ and

$$J(t, u) = \max(\|u\|_X, t \|u\|_Y).$$

Since Γ is assumed to be bounded on Y we have $N(t, \Gamma) \leqslant C/t$, for $t \geqslant 1$. Note also that if $\Gamma \neq 0$ then

$$N(t, \Gamma)^{-1} = \inf\{J(t, w) : w \in Y, \Gamma(w) = 1\}. \tag{1}$$

To see this we first note that $|\Gamma(w)| \leqslant N(t, \Gamma)\, J(t, w)$ which gives half of (1). For the other half we put $w = u/\Gamma(u)$ if $\Gamma(u) \neq 0$. Then $\Gamma(w) = 1$ and $J(t, w) = J(t, u)/|\Gamma(u)|$, which gives the remaining half.

Theorem 1. *Let Γ be a bounded linear functional on Y. Then Γ can be extended to a bounded linear functional on the real interpolation space $(X, Y)_{\theta,p}$, if and only if*

$$N_{\theta,p}(\Gamma) = \left(\sum_{k=0}^{\infty} (2^{-k\theta} N(2^{-k}, \Gamma))^{p'} \right)^{1/p'} < \infty. \tag{2}$$

Here $1/p' = 1 - 1/p$ *and* $0 < \theta < 1$, $1 \leqslant p \leqslant \infty$ *or* $0 \leqslant \theta \leqslant 1$, $p = \infty$. *The norm of the extended functional is equivalent to* $N_{\theta,p}(\Gamma)$.

Remark. If Y is dense in X then $N(1/t, \Gamma)$ is equal to the K-functional for the duals of X' and Y' of X and Y. If in addition $p < \infty$ the theorem is equivalent to the so called *duality theorem* of interpolation theory. (See [1], section 3.7). In this case condition (2) means that $\Gamma \in (X', Y')_{\theta,p'}$.

Proof. Take $x \in (X, Y)_{\theta,p}$. Then we can write $x = \sum_{k=0}^{\infty} u_k$ where $u_k \in Y$ and

$$\left(\sum_{k=0}^{\infty} (2^{k\theta} J(2^{-k}, u_k))^p\right)^{1/p} \leqslant C \|x\|_{\theta,p}.$$

If $p < \infty$ we have that

$$\left\| x - \sum_{k=0}^{n} u_k \right\|_{\theta,p} \leqslant C \left(\sum_{k>n} (2^{k\theta} J(2^{-k}, u_k))^p\right)^{1/p} \to 0, \tag{3}$$

as $n \to \infty$. Moreover

$$\sum_{k=0}^{\infty} |\Gamma(u_k)| \leqslant \sum_{k=0}^{\infty} N(2^{-k}, \Gamma)\, J(2^{-k}, u_k) \leqslant C N_{\theta,p}(\Gamma)\, \|x\|_{\theta,p}.$$

We now define $\Gamma(x)$ by the formula $\Gamma(x) = \sum_0^{\infty} \Gamma(u_k)$. Then Γ is unambigously defined on $(X, Y)_{\theta,p}$. In fact, if $x = \sum_0^{\infty} w_k$ is another representation of x we use (3) and the estimate above to deduce that

$$\left| \sum_{k=0}^{n} \Gamma(u_k) - \sum_{k=0}^{n} \Gamma(w_k) \right| \leqslant C N_{\theta,p}(\Gamma) \left\| \sum_{k=0}^{n} (u_k - w_k) \right\|_{\theta,p} \to 0.$$

Clearly Γ is now extended to a bounded linear functional on $(X, Y)_{\theta,p}$ with norm bounded by a constant times $N_{\theta,p}(\Gamma)$.

In the case $p = \infty$, we can extend Γ to the closure of Y in the space $(X, Y)_{\theta,\infty}$ using the same construction as above. Then we can use the Hahn-Banach theorem to extend Γ to the full interpolation space.

To prove the converse implication, assume that Γ can be extended to $(X, Y)_{\theta,p}$ with norm M. For given positive numbers a_k we can choose $u_k \in Y$ so that $J(2^{-k}, u_k) = a_k$ and $N(2^{-k}, \Gamma)\, a_k \leqslant 2\Gamma(u_k)$. Since $x_n = \sum_{k=0}^{n} u_k$ belongs to $(X, Y)_{\theta,p}$ we have

$$\Gamma(x_n) \leqslant M \|x_n\|_{\theta,p} \leqslant CM \left(\sum_{k=0}^{n} (2^{k\theta} a_k)^p\right)^{1/p}.$$

We conclude that

$$\sum_{k=0}^{n} N(2^{-k}, \Gamma) a_k \leqslant CM \left(\sum_{k=0}^{n} (2^{k\theta} a_k)^p \right)^{1/p},$$

and hence $N_{\theta,p}(\Gamma) \leqslant CM$. This completes the proof.

COROLLARY 1. *Suppose that* $N(t, \Gamma) = \mathcal{O}(t^{-\theta_0})$ *as* $t \to 0$. *Then* Γ *can be extended to a bounded linear functional on* $(X, Y)_{\theta,p}$ *for all* $\theta > \theta_0$.

The proof is immediate.

3. INTERPOLATION WITH CONSTRAINTS

Assume that Y is continuously embedded in X and let $\mathcal{G}$ be a finite set of bounded linear functionals on Y. Then we let $Y_{\mathcal{G}}$ be the subspace of Y consisting of all $u \in Y$ satisfying the *constraints* $\Gamma(u) = 0$, for all $\Gamma \in \mathcal{G}$. We shall now investigate the relation between the interpolation spaces $(X, Y_{\mathcal{G}})_{\theta,p}$ and $(X, Y)_{\theta,p}$. Since $Y_{\mathcal{G}}$ is a subspace of Y we have $(X, Y_{\mathcal{G}})_{\theta,p} \subset (X, Y)_{\theta,p}$. Thus the converse inclusion is at focus in the sequel.

We shall define a *strongly independent basis* for the set $\mathcal{G}$. If $\mathcal{H}$ is any subset of $\mathcal{G}$ we put

$$N_{\mathcal{H}}(t, \Gamma) = \sup\{ |\Gamma(u)| : J(t, u) \leqslant 1 \text{ and } \Delta(u) = 0 \text{ of all } \Delta \in \mathcal{H} \}.$$

Obviously we have that

$$N_{\mathcal{H}}(t, \Gamma) \geqslant N_{\mathcal{K}}(t, \Gamma) \qquad \text{if} \quad \mathcal{H} \subset \mathcal{K}. \tag{1}$$

DEFINITION 1. If $\{\Gamma_1, ..., \Gamma_M\}$ is a given set of functionals we put $N_m = N_{\mathcal{H}_m}$ where $\mathcal{H}_m = \{\Gamma_1, ..., \Gamma_m\}$ and $m = 1, ..., M$. We also write $N_0 = N$.

The functionals $\Gamma_1, ..., \Gamma_M$ are called strongly independent if for all m we have that $N(t, \Gamma_m) = \mathcal{O}(N_{m-1}(t, \Gamma_m))$ as $t \to 0$. We then say that Γ_m has order θ_m, if $N(t, \Gamma_m) \sim t^{-\theta_m}$ i.e. if there are positive constants c_1 and c_2 such that

$$c_1 t^{-\theta_m} \leqslant N(t, \Gamma) \leqslant c_2 t^{-\theta_m}$$

for all sufficiently small values of t.

We say that $\Gamma_1, ..., \Gamma_M$ is a strongly independent basis for the set $\mathcal{G}$ if $\Gamma_1, ..., \Gamma_M$ are strongly independent and form a linear basis for the space spanned by the functionals in $\mathcal{G}$.

There is a technically useful equivalent formulation of the definition of strong independence which we give in our next definition.

DEFINITION 2. The elements $w_1, ..., w_M \in Y$ (depending on t) form a supporting sequence for the functionals $\Gamma_1, ..., \Gamma_M$, if there exist numbers $A > 0$ and $t_0 > 0$ such that

$$\Gamma_r(w_k) = \delta_{r,k}, \qquad J(t, w_k) \leqslant A/N(t, \Gamma_k), \tag{2}$$

for $0 < t < t_0$ and for $r, k = 1, ..., M$.

LEMMA 1. *The functionals $\Gamma_1, ..., \Gamma_M$ are strongly independent if and only if there exist a supporting sequence $w_1, ..., w_M \in Y$.*

Proof. Using the definition of $N_{k-1}(t, \Gamma_k)$ we see that for all $k = 1, ..., M$ there exists $v_k \in Y$ such that

$$\Gamma_r(v_k) = \delta_{r,k} \qquad \text{for all} \quad r \leqslant k \tag{3}$$

$$J(t, v_k) \leqslant A_k/N_{k-1}(t, \Gamma_k) \tag{4}$$

Now we define w_m recursively by means of the formulas

$$w_M = v_M, \qquad w_m = v_m - \sum_{k=m+1}^{M} \Gamma_k(v_m) w_k.$$

Then it is clear from (3) that $\Gamma_r(w_k) = \delta_{r,k}$ for all r and k. Moreover, using the definition of N_{m-1} and (4), we get

$$\begin{aligned} J(t, w_m) &\leqslant J(t, v_m) + \sum_{k=m+1}^{M} |\Gamma_k(v_m)|\, J(t, w_k) \\ &\leqslant J(t, v_m) \cdot \left(1 + \sum_{k=m+1}^{M} N_{m-1}(t, \Gamma_k)\, J(t, w_k)\right). \end{aligned}$$

Assuming that we have already proved that $J(t, w_k)$ is bounded by a constant divided by $N_{k-1}(t, \Gamma_k)$ for $k = m+1, ..., M$, we now get

$$J(t, w_m) \leqslant CJ(t, v_m) \left(1 + \sum_{k=m+1}^{M} \frac{N_{m-1}(t, \Gamma_k)}{N_{k-1}(t, \Gamma_k)}\right)$$

Using the assumption we have that $N_{k-1}(t, \Gamma_k) \geqslant CN(t, \Gamma_k)$. In view of (1) we have $N_{m-1} \leqslant N$. Therefore we can conclude that

$$\frac{N_{m-1}(t, \Gamma_k)}{N_{k-1}(t, \Gamma_k)}$$

is bounded for all small values of t. From the assumption on v_m therefore we get that

$$J(t, w_m) \leqslant CJ(t, v_m) \leqslant C'/N(t, \Gamma_m).$$

This implies the first part of the lemma.

To prove the second part we have only to observe that

$$1/N_{m-1}(t, \Gamma_m) \leqslant J(t, w_m) \leqslant C/N(t, \Gamma_m).$$

Thus $N_{m-1}(t, \Gamma_m) \sim N(t, \Gamma_m)$. This completes the proof of the lemma.

We now give our first result on interpolation with constraints.

THEOREM 2. *Suppose that $\Gamma_1, ..., \Gamma_M$ is a strongly independent basis for the set $\mathscr{G}$ and that Γ_m has order θ_m, $(m = 1, ..., M)$. Then Γ_m can be extended to $(X, Y)_{\theta,p}$ if $\theta_m < \theta$. Moreover let θ be a given number such that $0 \leqslant \theta \leqslant 1$ and $\theta \notin \{\theta_1, ..., \theta_M\}$. Then we have the following conclusions:*

If $\theta < \theta_m$ for all m then $(X, Y_{\mathscr{G}})_{\theta,p} = (X, Y)_{\theta,p}$.
Otherwise, $(X, Y_{\mathscr{G}})_{\theta,p}$ consists of all $x \in (X, Y)_{\theta,p}$ such that $\Gamma_m(x) = 0$ for every m with $\theta_m < \theta$.

Proof. Using Corollary 1 we see that we can extend Γ_m to $(X, Y)_{\theta,p}$ if $\theta_m < \theta$. Writing $Y_m = Y_{\mathscr{K}_m}$ and $Y_0 = Y$, we shall use induction over m to show that

$$\theta < \theta_m \Rightarrow (X, Y_m)_{\theta,p} = (X, Y_{m-1})_{\theta,p} \tag{5}$$

$$\theta > \theta_m \Rightarrow (X, Y_m)_{\theta,p} = \{x \in (X, Y_{m-1})_{\theta,p} : \Gamma_m(x) = 0\}. \tag{6}$$

Here $m = 1, ..., M$. This will clearly give the theorem. First note that $(X, Y_m)_{\theta,p} \subset (X, Y_{m-1})_{\theta,p}$. In the case $\theta < \theta_m$ it is therefore enough to show the converse inclusion. Thus assume that $x \in (X, Y_{m-1})_{\theta,p}$. Then we can find $y_t \in Y$ such that $\Gamma_j(y_t) = 0$ for $j = 1, ..., m-1$ and

$$\|x - y_t\|_X + t\,\|y_t\|_Y \leqslant 2K(t, x; X, Y_{m-1}), \qquad 0 < t < \infty.$$

We choose $w \in Y$ so that $\Gamma_j(w) = 0$ for $j = 1, ..., m-1$, $\Gamma_m(w) = 1$ and $J(t, w) \leqslant A/N(t, \Gamma_m)$. This is possible according to Lemma 2 provided that now $0 < t < t_0$. Next we put $z = y_t - \Gamma_m(y_t)w$. Then it is clear that $z \in Y_m$. Therefore

$$K(t, x; X, Y_m) \leqslant \|x - z\|_X + t\,\|z\|_Y \leqslant 2(K(t, x; X, Y_{m-1}) + |\Gamma_m(y_t)|\, J(t, w))$$

In order to estimate $\Gamma_m(y_t)$ we put $v_n = y_{t2^{n+1}} - y_{t2^n}$. Since $y_{t2^n} \to 0$ in Y $(n \to \infty)$, we have that

$$|\Gamma_m(y_t)| \leqslant \sum_{n=0}^{\infty} |\Gamma_m(v_n)| \leqslant \sum_{n=0}^{\infty} N(t2^n, \Gamma_m)\, J(t2^n, v_n).$$

Now we observe that

$$J(t2^n, v_n) \leqslant 2K(t2^n, x; X, Y_{m-1}).$$
$$N(t2^n, \Gamma_m) \leqslant CN(t, \Gamma_m)\, 2^{-n\theta_m}$$

Therefore we get the estimate

$$K(t, x; X, Y_m) \leqslant C \sum_{n=0}^{\infty} 2^{-n\theta_m} K(t2^n, x; X, Y_{m-1}),$$

which implies that $x \in (X, Y_m)_{\theta,p}$ if $\theta < \theta_m$. This proves (5). Next consider the case $\theta > \theta_m$. Assume that $x \in (X, Y_{m-1})_{\theta,p}$ and that $\Gamma_m(x) = 0$. Then we choose y_t as above but this time we consider v_{-n}. Since $y_{t2^{-n}} \to x$ in X we now get

$$|\Gamma_m(y_t)| = |\Gamma_m(x - y_t)| \leqslant \sum_{n=0}^{\infty} |\Gamma_m(v_{-n})|.$$

In the same way as above we deduce

$$K(t, x; X, Y_m) \leqslant C \sum_{n=0}^{\infty} 2^{n\theta_m} K(t2^{-n}, x; X, Y_{m-1}),$$

for all small values of t. This implies that $x \in (X, Y_m)_{\theta,p}$ if $\theta > \theta_m$.

Conversely, if $x \in (X, Y_m)_{\theta,p}$ and still $\theta > \theta_m$, we can write $x = \sum v_k$ where $v_k \in Y$ and $\Gamma_r(v_k) = 0$ for $k = 1, ..., m$. Since Γ_m is bounded on $(X, Y)_{\theta,p}$ and $\sum v_k$ converges to x there, we conclude that $\Gamma_m(x) = 0$. This proves (6). The proof of theorem 2 is now complete.

We shall now consider the excluded case $\theta \in \{\theta_1, ..., \theta_M\}$ in Theorem 2. We first need a definition.

DEFINITION 3. We shall say that the couple (X, Y) is linearizable by means of the family $\Lambda(t)$, $0 < t < 1$, if $\Lambda(t)$ is strongly continuous in t on X and

$$K(t, x; X, Y) \sim \|x - \Lambda(t)x\|_X + t\, \|\Lambda(t)x\|_Y.$$

It is easy to see that (X, Y) is linearizable by means of $\Lambda(t)$, if and only if $\max(\|x - \Lambda(t)x\|_X, t\, \|\Lambda(t)x\|_Y) \leqslant C \min(\|x\|_X, t\, \|x\|_Y)$.

THEOREM 3. *Suppose that $\Gamma_1, \ldots, \Gamma_M$ is a strongly independent basis for the set $\mathscr{G}$ and that Γ_m has order θ_m, $(m = 1, \ldots, M)$. Assume moreover that the couple (X, Y) is linearizable by means of the family $\Lambda(t)$. Then the space $(X, Y_{\mathscr{G}})_{\theta,p}$ consists of all $x \in (X, Y)_{\theta,p}$ such that*

$$\Gamma_m(x) = 0, \qquad \text{for all } m \text{ with } \quad \theta_m < \theta, \tag{7}$$

$$\left(\int_0^1 |\Gamma_m(\Lambda(t)x)|^p \frac{dt}{t}\right)^{1/p} < \infty \qquad \text{for all } m \text{ with } \quad \theta_m = \theta. \tag{8}$$

Proof. According to Lemma 1 we can choose a supporting sequence (w_k). Put

$$\Lambda_m(t)x = \Lambda(t)x - \sum_{k=1}^{m} \Gamma_k(\Lambda(t)w)w_k, \qquad m = 1, \ldots, M.$$

Using the notation of the proof of Theorem 2, we shall now prove that $x \in (X, Y_m)_{\theta,p}$ if and only if $x \in (X, Y_{m-1})_{\theta,p}$ and

$$\left(\int_0^1 \left(\frac{|\Gamma_m(\Lambda(t)x)|}{t^\theta N(t, \Gamma_m)}\right)^p \frac{dt}{t}\right)^{1/p} < \infty. \tag{9}$$

This will give the result in view of Theorem 2.

First we note that Λ_m maps X into Y_m. Using the assumption on $\Lambda(t)$ it is easy to see that if $x \in Y$ and $\Gamma_m(x) = 0$ then

$$|\Gamma_m(\Lambda(t)x)| \leqslant CN(t, \Gamma_m) \min(\|x\|_X, t\,\|x\|_Y). \tag{10}$$

This implies in particular that (X, Y_m) is linearizable by means of $\Lambda_m(t)$. Writing $\Lambda_0(t) = \Lambda(t)$ we have the recursive formula

$$\Lambda_m(t)x = \Lambda_{m-1}(t)x - \Gamma_m(\Lambda(t)x)w_m$$

Then the assumptions imply

$$K(t, x; X, Y_m) \leqslant C\left(K(t, x; X, Y_{m-1}) + \frac{|\Gamma_m(\Lambda(t)x)|}{N(t, \Gamma_m)}\right).$$

Thus if $x \in (X, Y_{m-1})_{\theta,p}$ and (9) holds we can conclude that $x \in (X, Y_m)_{\theta,p}$. Conversely, if $x \in (X, Y_m)_{\theta,p}$ we can use (10) to get the estimate

$$|\Gamma_m(\Lambda(t)x)| \leqslant CN(t, \Gamma_m)\, K(t, x; X, Y_m).$$

This implies (9). The proof of Theorem 3 is complete.

COROLLARY 2. *Suppose that $\Gamma_1, \ldots, \Gamma_M$ is a strongly independent basis for the set $\mathscr{G}$ and that Γ_m has order θ_m, $(m = 1, \ldots, M)$. Assume moreover that* *the couple (X, Y) is linearizable by means of the family $A(t)$. Then the couple $(X, Y_{\mathscr{G}})$ is linearizable by means of the family*

$$A_{\mathscr{G}}(t)x = A(t)x - \sum_{k=1}^{M} \Gamma_k(A(t)x)w_k.$$

Here $w_1, ..., w_M$ is a supporting sequence for $\Gamma_1, ..., \Gamma_M$.

4. Perturbation of Functionals

In the theorems of the previous section it is important to have a strongly independent base for the set $\mathscr{G}$ of linear functionals defining the constraints. We shall here give some results that are useful to prove strong independence. One idea is to consider the functionals in $\mathscr{G}$ as perturbations of simpler functionals which are known to be strongly independent. We start however with a simple sufficient condition for strong independence.

Lemma 2. *Let $\Gamma_1, ..., \Gamma_M$ be bounded linear functionals on Y. Assume that there exist $u_1, ..., u_M$, depending on t, and positive constants A and B, such that*

$$\Gamma_k(u_k) = 1, \qquad J(t, u_k) \leqslant A/N(t, \Gamma_k), \qquad k = 1, ..., M, \tag{1}$$

$$|\det[\Gamma_r(u_k)]| \geqslant B, \tag{2}$$

for $0 < t < 1$. Then $\Gamma_1, ..., \Gamma_M$ are strongly independent.

The same conclusion holds if (2) is replaced by the condition

$$\lim_{t \to 0} \Gamma_r(u_k) \cdot \frac{N(t, \Gamma_k)}{N(t, \Gamma_r)} = a_{r,k}, \qquad \text{where} \quad \det[a_{r,k}] \neq 0. \tag{3}$$

Proof. For each m we put $w_m = \sum_{k=1}^{M} c_{m,k}u_k$, where $c_{m,k}$ are the solutions of the system

$$\sum_{k=1}^{M} c_{m,k}\Gamma_r(u_k) = \delta_{m,r}, \qquad r = 1, ..., M.$$

Then $\Gamma_r(w_m) = \delta_{m,r}$. We need to estimate $J(t, w_m)$. Let π run through all permutations of $1, ..., M$ and put $D = |\det[\Gamma_r(u_k)]|$. Then

$$\begin{aligned} |c_{m,k}| &\leqslant D^{-1} \sum_{\pi_\nu \neq k} \prod_{\nu \neq m} |\Gamma_\nu(u_{\pi_\nu})| \leqslant D^{-1} \sum_{\pi_\nu \neq k} \prod_{\nu \neq m} N(t, \Gamma_\nu)\, J(t, u_{\pi_\nu}) \\ &\leqslant A^{M-1}B^{-1} \sum_{\pi_\nu \neq k} \prod_{\nu \neq m} N(t, \Gamma_\nu)/N(t, \Gamma_{\pi_\nu}). \end{aligned}$$

Thus we get the estimate

$$|c_{m,k}| \leqslant (M-1)!\, A^{M-1} B^{-1} N(t, \Gamma_k)/N(t, \Gamma_m).$$

Therefore we conclude that

$$J(t, w_m) \leqslant \sum_{k=1}^{M} |c_{m,k}|\, J(t, u_k) \leqslant M!\, A^M B^{-1}/N(t, \Gamma_m)$$

This proves the first part of the lemma. The second part is obvious, since

$$\det[\Gamma_r(u_k)] = \det\left[\Gamma_r(u_k) \cdot \frac{N(t, \Gamma_k)}{N(t, \Gamma_r)}\right] \to \det[a_{r,k}] = 0, \qquad \text{as} \quad t \to 0.$$

The proof is complete.

DEFINITION 4. Let Γ and Δ be two bounded linear functionals on Y. Then we say that Γ dominates Δ if

$$\frac{N(t, \Delta)}{N(t, \Gamma)} \to 0 \qquad \text{as} \quad t \to 0.$$

LEMMA 3. *Let $\tilde{\Gamma}_1, ..., \tilde{\Gamma}_M$ be strongly independent functionals and suppose that $\Gamma_m = \tilde{\Gamma}_m + \Delta_m$, where $\tilde{\Gamma}_m$ dominates Δ_m, $(m = 1, ..., M)$. Then $\Gamma_1, ..., \Gamma_M$ are strongly independent and*

$$N(t, \Gamma_m) \sim N(t, \tilde{\Gamma}_m), \qquad m = 1, ..., M.$$

Proof. By assumption we can find a supporting sequence $w_1, ..., w_M$ for $\tilde{\Gamma}_1, ..., \tilde{\Gamma}_M$. Put $u_k = a_k w_k$, where $\alpha_k = 1/(1 + \Delta_k(w_k))$. Then $\Gamma_k(u_k) = 1$. Note also that

$$|\Delta_k(w_k)| \leqslant N(t, \Delta_k)\, J(t, w_k) \leqslant AN(t, \Delta_k)/N(t, \tilde{\Gamma}_k).$$

Since $\tilde{\Gamma}_k$ dominates Δ_k we conclude that $\Delta_k(w_k) \to 0$ as $t \to 0$. Thus $\alpha_k \to 1$ and therefore $|\alpha_k| \leqslant 2$ for small values of t. Consequently

$$J(t, u_k) = |\alpha_k|\, J(t, w_k) \leqslant 2A/N(t, \tilde{\Gamma}_k).$$

It follows that $N(t, \tilde{\Gamma}_k) \leqslant 2AN(t, \Gamma_k)$. Conversely we have for small values of t that

$$N(t, \Gamma_k) \leqslant N(t, \tilde{\Gamma}_k) + N(t, \Delta_k) \leqslant 2N(t, \tilde{\Gamma}_k).$$

Therefore $N(t, \Gamma_k) \sim N(t, \tilde{\Gamma}_k)$ and thus $J(t, u_k) \leqslant C/N(t, \Gamma_k)$. Moreover

$$\det[\Gamma_r(u_k)] = \prod_{k=1}^{M} \alpha_k \cdot \det\left[\delta_{r,k} + \Delta_r(w_k) \cdot \frac{N(t, \tilde{\Gamma}_k)}{N(t, \tilde{\Gamma}_r)}\right]$$

Since

$$|\Delta_r(w_k)| \cdot \frac{N(t, \tilde{\Gamma}_k)}{N(t, \tilde{\Gamma}_r)} \leqslant A \frac{N(t, \Delta_r)}{N(t, \tilde{\Gamma}_r)} \to 0$$

we have that

$$\lim_{t \to 0} \det[\Gamma_r(u_k)] = 1.$$

This implies that $\Gamma_1, \ldots, \Gamma_M$ are strongly independent. The proof is complete.

Example. We conclude with an example to illustrate the importance of the concept of strongly independent basis. Let us start with two functionals A and B. Assume that $N(t, A) \sim t^{-\alpha}$ and $N(t, B) \sim t^{-\beta}$, where $\alpha > \beta$. Moreover assume that A, B are strongly independent. Then let $\mathscr{G}$ be the set $\{A + B, A\}$. By lemma 3 (used with just one functional) we have that $N(t, A + B) \sim t^{-\alpha}$. However, $A + B$ and A are not strongly independent. In fact, assume that they were. Then we could find u and v so that $(A + B)(u) = A(v) = 1$, $A(u) = (A + B)(v) = 0$, $J(t, u) \leqslant Ct^{\alpha}$ and $J(t, v) \leqslant Ct^{\alpha}$. This gives the contradiction

$$1 = B(u) \leqslant C_1 N(t, B) t^{\alpha} \leqslant C_2 t^{\alpha - \beta}.$$

However, A, B is a strongly independent basis for $\mathscr{G}$. Note also that theorem 2, incorrectly used with the functionals $A + B$, B, would not give the correct breakpoints α, β.

Proposition. *Suppose that $\Gamma_1, \ldots, \Gamma_M$ and $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_M$ are two strongly independent bases for $\mathscr{G}$. Let θ_m and $\tilde{\theta}_m$ be the orders of Γ_m and $\tilde{\Gamma}_m$, respectively, $(m = 1, \ldots, M)$. Then the two sets of breakpoints $\{\theta_1, \ldots, \theta_M\}$ and $\{\tilde{\theta}_1, \ldots, \tilde{\theta}_M\}$ are equal.*

Proof. It is no restriction to assume that $\theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_M$ and that $\tilde{\theta}_1 \leqslant \tilde{\theta}_2 \leqslant \cdots \leqslant \tilde{\theta}_M$. Now we can write

$$\tilde{\Gamma}_M = \lambda_M \Gamma_M + \sum_{k=1}^{M-1} \lambda_k \Gamma_k$$

It follows that $N(t, \tilde{\Gamma}_M) \leqslant Ct^{-\theta_M}$. Changing the roles of $\tilde{\Gamma}_M$ and Γ_M we also get $N(t, \Gamma_M) \leqslant Ct^{-\tilde{\theta}_M}$. Thus $\theta_M = \tilde{\theta}_M$. Excluding the functionals Γ_M and $\tilde{\Gamma}_M$ from the set $\mathscr{G}$ and proceeding in the same way on the remaining sequences we get the conclusion.

5. Interpolation of Sobolev Spaces with Constraints

We shall here consider Sobolev spaces of functions in one variable. To be specific we work with functions defined on the one-dimensional torus $\mathbb{T}$. Thus let X denote the Lebesgue space $L_p(\mathbb{T})$ and let Y denote the Sobolev space of all $u \in X$ such that the N:th derivative $D^N u \in X$. As norm on Y we use

$$\|u\|_Y = \|u\|_X + \|D^N u\|_X.$$

The interpolation spaces $(X, Y)_{\theta,p}$ are the well known Besov spaces $B^s_{p,p}$ where $s = N\theta$. We shall consider functionals of the general form

$$\Gamma(u) = (D^n u)(s_0) + \sum_{k<n} \int_{\mathbb{T}} (D^k u)\, dv_k \tag{1}$$

where s_0 is a given point and v_k are bounded measures on $\mathbb{T}$. The functional defined by the sum on the right hand side will be treated as perturbation of the first term.

Lemma 4. *Put $\tilde{\Gamma}(u) = (D^n u)(s_0)$ where $n + 1/p < N$. Then $N(t, \tilde{\Gamma}) \sim t^{-\theta_n}$, where $\theta_n = (n + 1/p)/N$. Moreover, if Γ is defined by (1), then $\tilde{\Gamma}$ dominates $\Delta = \Gamma - \tilde{\Gamma}$ and $N(t, \Gamma) \sim t^{-\theta_n}$.*

Proof. Using Sobolev's embedding theorem if $p < \infty$ and Kolmogorov's inequality if $p = \infty$, we get that

$$|D^n u(s)| \leqslant C\, \|u\|_X^{1-\theta_n} \|u\|_Y^{\theta_n} \leqslant Ct^{-\theta_n} J(t, u). \tag{2}$$

Thus $N(t, \tilde{\Gamma}) \leqslant Ct^{-\theta_n}$.

The converse inequality calls for a construction. Let ψ be a given infinitely differentiable function on the real line such that $\psi(\xi) = 1$, $|\xi| \leqslant 1/2$ and $\psi(\xi) = 0$, $|\xi| \geqslant 1$. Then we put

$$\Phi_t(s) = \sum_{v=-\infty}^{\infty} \psi((\tau v)^N) \exp(-ivs), \qquad \text{where} \quad \tau = t^{1/N}. \tag{3}$$

We can choose ψ so that Φ_t is non-negative. The proof of the following estimate is left to the reader:

$$|D^n\Phi_t(s)| \leqslant C_{k,n} \frac{\tau^{-n-1}}{1+(|\sin(s)|/\tau)^{2k}}, \qquad n, k = 0, 1, 2, \dots \tag{4}$$

Since Φ_t is non-negative we have

$$\int_{\mathbb{T}} \Phi_t(s)\, d\mu(s) = 1, \qquad \text{where} \quad d\mu(s) = \frac{ds}{2\pi}. \tag{5}$$

Now we put

$$u = (\Phi_t * \bar{\varphi})/c^2,$$

where

$$\varphi(\sigma) = (D^n\Phi_t)(s_0 - \sigma), \qquad c^2 = \int_{\mathbb{T}} |\varphi(\sigma)|^2\, d\mu(\sigma)$$

We claim that

$$J(t, u) \leqslant Ct^{\theta_n}. \tag{6}$$

Since $\tilde{\Gamma}(u_n) = 1$ this implies that $N(t, \tilde{\Gamma}) \sim t^{-\theta_n}$. To prove (6) we first use Bessel's equality to estimate c^2:

$$c^2 = \tau^{-2n-1} \sum_{\nu} |(i\tau\nu)^n\, \psi((\tau\nu)^N)|^2\, \tau$$

Here the sum can be considered as a Riemann sum for $\int_{\mathbb{R}} |(i\xi)^n\, \psi(\xi^N)|^2\, d\xi$. Thus we conclude that

$$c^2 \sim \tau^{-2n-1}$$

Now (3) and (4) imply that

$$\|\Phi_t * \bar{\varphi}\|_X \leqslant \|\bar{\varphi}\|_X \leqslant C\tau^{-n-1+1/p},$$

$$\|\Phi_t * \bar{\varphi}\|_Y \leqslant C\tau^{-N}\, \|\bar{\varphi}\|_X \leqslant C\tau^{-N-n-1+1/p}.$$

Thus we get

$$J(t, u) \leqslant C\tau^{-n-1+1/p}/c_n^2 = Ct^{\theta_n}$$

proving (6).

Now we have proved the first part of the lemma. To prove the second half it is enough to consider the case $\Delta(x) = \int_{\mathbb{T}} (D^k x)\, dv$, where $k < n$ and v is a bounded measure. Now (2) implies that

$$|\Delta(x)| \leqslant Ct^{-\theta_k} J(t, x).$$

Thus $N(t, \Delta) \leqslant Ct^{-\theta_k}$. Since $\theta_k < \theta_n$ we conclude that $\tilde{\Gamma}$ dominates Δ. The proof of the lemma is complete.

We shall now consider sets of functionals of the form (1). Using the previous lemma we can however concentrate on functionals defined by pure point evaluations of derivatives. To begin with we consider point evaluations at one point s_0.

LEMMA 5. *Consider functionals of the form*

$$\Gamma_k(u) = (D^{n_k} u)(s_0), \qquad k = 1, \ldots, K$$

where $n_k \neq n_r$ if $k \neq r$. Then $\Gamma_1, \ldots, \Gamma_K$ are strongly independent.

Proof. We base the proof on the construction of the previous lemma. Thus we put $u_k = (\Phi_t * \overline{\varphi_k})/c_k^2$ where $\varphi_k(\sigma) = (D^{n_k}\Phi_t)(s_0 - \sigma)$ and $c_k^2 \sim \tau^{-2n_k - 1}$. Then we have $\Gamma_k(u_k) = 1$ and $J(t, u_k) \leqslant C/N(t, \Gamma_k)$. Using lemma 2 we have only to prove that

$$\lim_{t \to 0} \Gamma_r(u_k)\, t^{\theta_r - \theta_k} = a_{r,k}, \qquad \text{where} \quad \det[a_{r,k}] \neq 0.$$

This is easily done using Parsevals formula and approximation by Riemann sums in the following way:

$$\Gamma_r(u_k)\, t^{\theta_r - \theta_k} = \frac{\int \tau^{n_r}\varphi_r(\sigma)\, \tau^{n_k}\overline{\varphi_k(\sigma)}\, d\mu(\sigma)}{\int |\tau^{n_k}\varphi_k(\sigma)|^2\, d\mu(\sigma)}$$
$$= \frac{\sum (i\tau v)^{n_r + n_k}\, |\psi((\tau v)^N)|^2\, \tau}{\sum |(i\tau v)^{n_k}\, \psi((\tau v)^N)|^2\, \tau} \to \frac{c_{r,k}}{b_k}$$

where

$$c_{r,k} = \int_{\mathbb{R}} (i\tau\xi)^{n_r + n_k}\, |\psi(\xi^N)|^2\, d\xi, \qquad b_k = \int_{\mathbb{R}} |(i\xi)^{n_k}\, \psi(\xi^N)|^2\, d\xi.$$

It is obviously enough to prove that $\det[c_{r,k}] \neq 0$. To see that let $z_1, \ldots, z_K$ be arbitrary complex numbers such that $\sum |z_k|^2 = 1$. Then we consider the quadratic form

$$Q = \sum_{r,k} c_{r,k} z_r \overline{z_k} = \int_{\mathbb{R}} \left| \sum_k (i\xi)^{n_k} z_k \right|^2 |\psi(\xi^N)|^2\, d\xi.$$

If not two of the integers $n_1, \ldots, n_K$ are equal then $\sum (i\xi)^{n_k} z_k \neq 0$ for almost all ξ and all $z_1, \ldots, z_K$. Thus the quadratic form Q is positive and therefore $\det[c_{r,k}] \neq 0$. This completes the proof.

We shall now consider point evaluations at different points.

LEMMA 6. *Consider functionals $\Gamma_{j,k}$ of the form*

$$\Gamma_{j,k}(u) = (D^{n_{j,k}}u)(s_j), \qquad j = 1, \ldots, J, \qquad k = 1, \ldots, K_j,$$

where $s_i \neq s_j$ if $i \neq j$ and $n_{j,k} \neq n_{j,r}$ if $k \neq r$. Then the set $\{\Gamma_{j,k}\}$ is strongly independent.

Proof. For each fixed j we have a set $\Gamma_{j,k}$ of strongly independent functionals. By lemma 1 we can find a supporting sequence $w_{j,k}$, $k = 1, \ldots, K_j$. Let χ_j be an infinitely differentiable function such that $\chi_j = 1$ in a small neighbourhood of s_j and $\chi_j = 0$ in the neighbourhood of all s_i, $i \neq j$. Put $\tilde{w}_{j,k} = \chi_j w_{j,k}$. Then it is easy to see that $\tilde{w}_{j,k}$ is a supporting sequence for the full set $\{\Gamma_{j,k}, j = 1, \ldots, J, k = 1, \ldots, K_j\}$. This proves the lemma.

THEOREM 4. *Let $\mathscr{G}$ be the set of all functionals defined for $j = 1, \ldots, J$ and $k = 1, \ldots, K_j$ by the equation*

$$\Gamma_{j,k}(u) = (D^{n_{j,k}}u)(s_j) + \sum_{r < n_{j,k}} \int_{\mathbb{T}} (D^r(u))\, dv_{j,k,r}. \tag{7}$$

Here $v_{j,k,r}$ are bounded measures on $\mathbb{T}$. Moreover $s_1, \ldots, s_J$ are distinct points, $n_{j,k} \neq n_{j,r}$ if $r \neq k$ and $n_{j,k} + 1/p < N$. Put $\theta_{j,k} = (n_{j,k} + 1/p)/N$. Then the interpolation space $(X, Y_{\mathscr{G}})_{\theta,p}$ consists of all $x \in (X, Y)_{\theta,p}$ such that

$$\Gamma_{j,k}(x) = 0, \qquad \textit{for all } j, k \textit{ for which} \quad \theta_{j,k} < \theta \tag{8}$$

$$\left(\int_0^1 \left(\frac{1}{t}\int_{-t}^{t} |\Gamma_{j,k}(s, x)|^p\, ds\right)^{p/p} \frac{d\tau}{\tau}\right)^{1/p} < \infty \qquad \textit{if} \quad \theta_{j,k} = \theta. \tag{9}$$

Here

$$\Gamma_{j,k}(s, x) = \Gamma_{j,k}(x(\bullet + s)).$$

Proof. From the previous lemmata we deduce that the functionals $\Gamma_{j,k}$ form a strongly independent basis for $\mathscr{G}$. Now define Φ_t by formula (3). Using the estimates in the proof of lemma 4 it is easy to see that (X, Y) is linearizable by means of the family $\Lambda(t)$ defined by $\Lambda(t)x = \Phi_t * x$. (We leave the details to the reader.) Using theorem 3 we see that it is sufficient to show that if $x \in (X, Y)_{\theta_{j,k},p}$ then (9) is equivalent to

$$\left(\int_0^1 |\Gamma_{j,k}(\Lambda(t)\, x)|^p \frac{dt}{t}\right)^{1/p} < \infty. \tag{10}$$

Note that

$$\Gamma_{j,k}(A(t)x) = (\Phi_t * \Gamma_{j,k}(\bullet, x))(0)$$

and that $\Gamma_{j,k}(\bullet, x) \in (X, Y)_{\eta,p}$, where $\eta = \theta_{j,k} - \eta_{j,k}/N = 1/(Np)$. Let W denote the Sobolev space of all $x \in X$ such that $Dx \in X$.
Then $(X, Y)_{\eta,p} = (X, W)_{1/p,p}$. Therefore it is sufficient to show that if $y \in (X, W)_{1/p,p}$ the following two conditions are equivalent:

$$\left(\int_0^1 |(\Phi_t * y)(0)|^p \frac{dt}{t}\right)^{1/p} < \infty, \tag{11}$$

$$\left(\int_0^1 \left(\frac{1}{t}\int_{-t}^{t} |y(s)|^p \, d\mu(s)\right)^{p/p} \frac{dt}{t}\right)^{1/p} < \infty. \tag{12}$$

First assume that $y \in (X, W)_{1/p,p}$ and that (11) holds. Let W_1 denote the sub-space of W of all u with $u(0) = 0$. Then Theorem 3 implies that $y \in (X, W_1)_{1/p,p}$. Then we can write $y = y_0 + y_1$ where $y_0 \in X$ and $y_1 \in W_1$ so that $y_1(0) = 0$. Let us put $\tilde{y}_1(s) = 0$ for $s < 0$ and $\tilde{y}_1(s) = y_1(s)$ for $s \geqslant 0$ and similarly for $\tilde{y}_0$ and $\tilde{y}$. Then it follows that $\tilde{y}_1 \in W$. Thus $\tilde{y} \in (X, W)_{1/p,p}$ implying that

$$\left(\int_0^1 \left(\frac{1}{t}\int_{\mathbb{T}} |\tilde{y}(s+t) - \tilde{y}(s)|^p \, ds\right)^{p/p} \frac{dt}{t}\right)^{1/p} < \infty$$

Restricting the domain of integration to the interval $-t < s < 0$ we get that

$$\left(\int_0^1 \left(\frac{1}{t}\int_0^t |y(s)|^p \, ds\right)^{p/p} \frac{dt}{t}\right)^{1/p} < \infty.$$

This proves half of (12). The other half follows by a symmetric construction. To prove that (12) implies (11) we write

$$(\Phi_t * y)(0) = \sum_{k=0}^{\infty} \int_{I_k} \Phi_t(-s)\, y(s)\, d\mu(s)$$

where

$$I_k = \{s : 2^{k-1}\tau \leqslant |s| < 2^k \tau\} \qquad \text{if} \quad 1 \leqslant k, \quad 2^k \tau \leqslant \pi,$$

$$I_0 = \{s : |s| < \tau\}$$

and I_k is empty for the remaining values of k. Then we use (4) and Hölder's inequality to get

$$\begin{aligned}
&\left|\int_{I_k} \Phi_t(-s)\, y(s)\, d\mu(s)\right| \\
&\quad\leqslant \left(\int_{I_k} |\Phi_t(-s)|^{p'}\, d\mu(s)\right)^{1/p'} \left(\int_{I_k} |y(s)|^p\, d\mu(s)\right)^{1/p} \\
&\quad\leqslant C\tau^{-1}\left(\int_{I_k} \frac{d\mu(s)}{(1+((|s|/\tau))^{2n})^{p'}}\right)^{1/p'} \left(\int_{I_k} |y(s)|^p\, d\mu(s)\right)^{1/p} \\
&\quad\leqslant C2^{(1-2n)k}\left(\frac{1}{\tau 2^k}\int_{I_k} |y(s)|^p\, d\mu(s)\right)^{1/p}
\end{aligned}$$

where $\tau = t^{1/N}$ and n is a fixed large number. Thus

$$|(\Phi_t * y)(0)| \leqslant C \sum_{\tau 2^k \leqslant \pi} 2^{(1-2n)k}\left(\frac{1}{\tau 2^k}\int_{|s| \leqslant \tau 2^k} |y(s)|^p\, d\mu(s)\right)^{1/p}$$

Now we note that

$$\begin{aligned}
&\left(\int_0^1 \left(\frac{1}{\tau 2^k}\int_{|s| \leqslant \tau 2^k} |y(s)|^p\, d\mu(s)\right)^{p/p} \frac{dt}{t}\right)^{1/p} \\
&\quad\leqslant C\left(\int_0^1 \left(\frac{1}{t}\int_{-t}^{t} |y(s)|^p\, d\mu(s)\right)^{p/p} \frac{dt}{t}\right)^{1/p}
\end{aligned}$$

Thus we conclude that (12) implies (11). This completes the proof.

Application to Best Approximation

Using Corollary 2 and the constructions of the preceeding proofs we get the following result, which is Jacksons inequality for approximation by trigonometric polynomials with constraints.

COROLLARY 3. *Under the assumptions of Theorem 5 the following holds. Let λ be a given positive integer. Assume that $x \in Y$ and $\Gamma_{j,k}(x) = 0$ for all j and k. Then there is a trigonometric polynomial Y of degree less than λ such that*

$$\|x - y\|_X \leqslant C\lambda^{-N}\, \|x\|_Y$$

and $\Gamma_{j,k}(y) = 0$ for all j and k.

Proof. Let $\Lambda(t)$ be defined as in the proof of the preceeding theorem. Let $\{w_{j,k}\}$ be a supporting sequence for the set $\mathscr{G} = \{\Gamma_{j,k}\}$. Using corollary 2 we see that the couple $(X, Y_{\mathscr{G}})$ is linearizable by means of the operators

$$\Lambda_{\mathscr{G}}(t)x = \Lambda(t)x - \sum_{j,k} \Gamma_{j,k}(\Lambda(t)x)\, w_{j,k}$$

By the proofs of lemma 4 and lemma 5 we see that $w_{j,k}$ are linear combinations of functions of the form $\Phi_t * v$. Thus $A_{\mathscr{G}}(t)x$ is a trigonometric polynomial of degree less than τ^{-1}, where $\tau = t^{1/N}$. Moreover we have that

$$\|x - A_{\mathscr{G}}(t)x\|_X \leqslant Ct\, \|x\|_Y$$

Writing $\tau = \lambda^{-1}$ we get the result.

Using Corollary 3 we can now deduce a theorem on best approximation with linear constraints. Let $E(\lambda, x)$ be the best approximation of x by means of trigonometric polynomials of degree less than λ, that is

$$E(\lambda, x) = \inf\{\, \|x - y\|_X : \deg(y) < \lambda\}$$

Similarly let $E_{\mathscr{G}}(\lambda, x)$ be the best approximation of x by means of trigonometric polynomials y satisfying the linear constraints $\Gamma(y) = 0$ for all Γ in the given set $\mathscr{G}$, i.e.

$$E_{\mathscr{G}}(\lambda, x) = \inf\{\, \|x - y\|_X : \deg(y) < \lambda,\ \Gamma(y) = 0 \text{ for all } \Gamma \in \mathscr{G}\}$$

COROLLARY 4. *Let $\mathscr{G}$ a the set of functionals satisfying the assumptions of theorem 5. Then for $0 < \alpha < N$ we have that*

$$\left(\sum_{n=0}^{\infty} (2^{-n\alpha})^p\, E_{\mathscr{G}}(2^n, x)^p \right)^{1/p} < \infty,$$

if and only if $x \in (X, Y)_{\alpha/N, p}$ and

$$\Gamma_{j,k}(u) = 0, \qquad \text{for all } j, k \text{ for which} \quad n_{j,k} < \alpha - 1/p$$

$$\left(\int_0^1 \left(\frac{1}{t} \int_{-t}^{t} |\Gamma_{j,k}(s, x)|^p\, ds \right)^{p/p} \frac{d\tau}{\tau} \right)^{1/p} < \infty \qquad \text{if} \quad n_{j,k} = \alpha - 1/p.$$

Here $\Gamma_{j,k}(s, x) = \Gamma_{j,k}(x(\bullet + s))$.

Proof. The result follows in a routine way from the previous corollary. We leave the details to the reader. See [6], section 3.

6. INTERPOLATION OF WEIGHTED L_p-SPACES

We shall here consider interpolation of weighted L_p-spaces. Let X denote the Lebesgue space $L_p(\mu)$ on a set Ω and $1 \leqslant p \leqslant \infty$. As space Y we take a weighted L_p-space with weight ω such that $\omega \geqslant 1$. The norm on Y is

$$\|u\|_Y = \|\omega u\|_X = \left(\int_\Omega (\omega\, |u|)^p\, d\mu \right)^{1/p}$$

We shall consider functionals of the form

$$\Gamma_\varphi(u) = \int_\Omega \varphi u \, d\mu$$

Clearly Γ_φ is bounded on Y is and only if

$$\left(\int_\Omega \left(\frac{|\varphi|}{\omega} \right)^q d\mu \right)^{1/q} < \infty \qquad \text{where} \quad \frac{1}{q} = 1 - \frac{1}{p}.$$

Note also that

$$J(t, u) \sim \left(\int_\Omega (\omega_t \, |u|)^p \, d\mu \right)^{1/p}$$

where $\omega_t = \max(1, t\omega)$. Therefore

$$N(t, \Gamma_\varphi) \sim \left(\int_\Omega \left(\frac{|\varphi|}{\omega_t} \right)^q d\mu \right)^{1/q}$$

We shall now give a general theorem, where strong independence is part of the assumptions. Then we consider three special situations where we prove strong independence. In the first case we consider what we call essentially disjoint functions φ. In the second case we discuss a general situation, but with the restriction $p = 2$. In the third case we consider general p but restrict ourselves to the real line and special choices of weight functions and functionals.

THEOREM 5. *Assume that $\varphi_1, ..., \varphi_M$ are given functions such that*

$$N_m(t) = \left(\int_{t\omega \geq 1} \left(\frac{|\varphi_m|}{t\omega} \right)^q d\mu \right)^{1/q} \sim t^{-\theta_m}, \qquad m = 1, ..., M, \tag{1}$$

where $0 < \theta_m < 1$. Define the corresponding functionals Γ_m by the formula

$$\Gamma_m(u) = \int \varphi_m u \, d\mu.$$

Assuming that these functionals are strongly independent we have the following result.

The interpolation space $(X, Y_{\mathscr{G}})_{\theta,p}$ consists of all $x \in (X, Y)_{\theta,p}$ such that

$$\Gamma_m(x) = 0 \qquad \textit{for all } m \textit{ with} \quad \theta_m < \theta, \tag{2}$$

$$\left(\int_0^1 \left| \frac{1}{t} \int_0^t \Gamma_m(\tau, x) \, d\tau \right|^p \frac{dt}{t} \right)^{1/p} < \infty \qquad \textit{if} \quad \theta_m = \theta. \tag{3}$$

Here

$$\Gamma_m(\tau, x) = \int_{\tau\omega < 1} \varphi_m x \, d\mu, \qquad 0 < \tau < 1.$$

Proof. First we note that condition (1) is equivalent to

$$N(t, \Gamma_m) \sim t^{-\theta_m} \tag{4}$$

This follows from the observation

$$N(t, \Gamma_m) \leqslant \left(\sum_{t \leqslant t2^n \leqslant 1} \int_{1 \leqslant 2^{-n}\omega < 2} |\varphi_m|^q \, d\mu + \int_{t\omega \geqslant 1} \left| \frac{\varphi_m}{t\omega} \right|^q d\mu \right)^{1/q}$$

Here the general term in the sum can be estimated by a constant times $N_m(2^{-n}) \leqslant C2^{n\theta_m}$. Thus (4) follows.

It is easy to see that the couple (X, Y) is linearizable by means of the family $\Lambda(t)$ defined by

$$\Lambda(t)x = \begin{cases} 0, & t\omega \geqslant 1 \\ x, & t\omega < 1 \end{cases}$$

Since $\Gamma_m(\Lambda(\tau)x) = \Gamma_m(\tau, x)$ the result now follows from theorem 3.

The Case with Essentially Disjoint Functions

DEFINITION 5. The functions $\psi_1, ..., \psi_M$ are said to be essentially disjoint if there are sets $\Omega_1, ..., \Omega_M$ such that ψ_k vanishes on Ω_r, $r \neq k$ and if

$$\left(\int_{t\omega \geqslant 1} \left| \frac{\psi_k}{t\omega} \right|^q d\mu \right)^{1/q} \leqslant C \left(\int_{\{t\omega \geqslant 1\} \cap \Omega_k} \left| \frac{\psi_k}{t\omega} \right|^q d\mu \right)^{1/q}$$

for some positive constant C and for all small values of t.

LEMMA 7. *Assume that $p > 1$ and that $\psi_1, ..., \psi_M$ are essentially disjoint and put*

$$\varphi_m = \psi_m + \chi_m$$

for $m = 1, ..., M$. Assume that

$$\left(\int_{t\omega \geqslant 1} \left| \frac{\psi_m}{t\omega} \right|^q d\mu \right)^{1/q} \sim t^{-\theta_m}$$

and

$$t^{\theta_m}\left(\int_{t\omega \geq 1}\left|\frac{\chi_m}{t\omega}\right|^q d\mu\right)^{1/q} \to 0$$

Then the corresponding functionals Γ_m are strongly independent. Moreover Γ_m has order θ_m.

Proof. We shall consider φ_m as perturbations of ψ_m. This is possible since the assumptions on χ_m means that the functional defined by χ_m is dominated by the functional defined by ψ_m. It is therefore enough to concentrate on the functions ψ_m.

Put $\Omega_m(t) = \{s: t\omega(s) \geq 1\} \cap \Omega_m$ and define u_m by the formula

$$u_m = \begin{cases} \psi_m^{-1}(|\psi_m|/t\omega)^q / \int_{\Omega_m(t)} (|\psi_m|/t\omega)^q \, d\mu & \text{on } \Omega_m(t) \\ 0 & \text{outside } \Omega_m(t) \end{cases}$$

Then $\tilde{\Gamma}_m(u_m) = 1$. Moreover $\tilde{\Gamma}_k(u_m) = 0$ if $k \neq m$ because ψ_k vanishes on Ω_m. Moreover

$$\begin{aligned} J(t, u_m) &\leq C\left(\int_{\Omega_m(t)}\left(\frac{|\psi_m|}{t\omega}\right)^{p(q-1)} d\mu\right)^{1/p} \Big/ \left(\int_{\Omega_m(t)}\left(\frac{|\psi_m|}{t\omega}\right)^q d\mu\right) \\ &\leq C\left(\int_{\Omega_m(t)}\left(\frac{|\psi_m|}{t\omega}\right)^q d\mu\right)^{-1/q} \end{aligned}$$

Using the assumptions we conclude that $\tilde{\Gamma}_m$ are strongly independent.

The Case $p = 2$

LEMMA 8. *Let $p = 2$ and assume that $\varphi_1, ..., \varphi_M$ are given functions. Define $N_m(t)$ by formula (1). Assume that for small values of t*

$$\int_{t\omega \geq 1}\left|\sum_{m=1}^{M}\frac{\varphi_m}{t\omega}\frac{z_m}{N_m(t)}\right|^2 d\mu \geq c_0 > 0 \qquad \text{if} \quad \sum_{m=1}^{M}|z_m|^2 = 1 \tag{5}$$

Then the corresponding functionals Γ_m, $m = 1, ..., M$ are strongly independent.

Proof. For each m we now put

$$u_m = \begin{cases} \overline{\varphi_m}/(t\omega N_m(t))^2, & t\omega \geq 1 \\ 0, & t\omega < 1 \end{cases}$$

Then $\Gamma_m(u_m) = 1$ and

$$J(t, u_m) \leq C\left(\int_{t\omega \geq 1}\left|\frac{\varphi_m}{t\omega}\right|^2 d\mu\right)^{1/2} \Big/ N_m(t)^2 \leq C/N_m(t)$$

It is now sufficient to prove that the determinant

$$\det[\Gamma_r(u_k)] = \det\left[\frac{N_k(t)\,\Gamma_r(u_k)}{N_r(t)}\right]$$

is bounded away from zero. To do that we consider the quadratic form

$$Q = \sum_{r,k} z_r \frac{N_k(t)\,\Gamma_r(u_k)}{N_r(t)} \bar{z}_k$$

Now Q is equal to the sum on the left of formula (5). Thus Q is bounded away from zero. It follows that $\Gamma_1, ..., \Gamma_M$ are strongly independent.

The Real Line, General p

We shall now consider the Lebesgue measure on the real line $\mathbb{R}$ with weightfunction $\omega(s) = |s|^a$ if $|s|$ is large and $\omega(s) \geqslant 1$. We shall consider functionals defined by functions φ which are perturbations of $\psi(s) = \omega(s)^\lambda$. We present our results in the following two lemmas.

LEMMA 9. *Assume that $\psi_m = \omega^{\lambda_m}$ for $m = 1, ..., M$. Put $\theta_m = \lambda_m + 1/aq$ where $1/q = 1 - 1/p$. Assume that the numbers λ_m are all different and that $0 < \theta_m < 1$. Let $\tilde{\Gamma}_m$ be defined by ψ_m so that*

$$\tilde{\Gamma}_m(u) = \int_{\mathbb{R}} \psi_m(s)\, u(s)\, ds$$

Then $\tilde{\Gamma}_1, ..., \tilde{\Gamma}_M$ are strongly independent and $\tilde{\Gamma}_m$ have order θ_m.

Proof. In order to prove the last formula it is enough to prove that

$$\left(\int_{t\omega \geqslant 1} \left(\frac{\psi_m}{t\omega}\right)^q ds\right)^{1/q} \sim t^{-\theta_m}.$$

This is however equivalent to

$$\frac{1}{t}\left(\int_{t\,|s|^a \geqslant 1} |s|^{a(\lambda_m - 1)q}\, ds\right)^{1/q} \sim t^{-\theta_m},$$

which is obvious. Now we introduce the functions

$$u_m = \begin{cases} \omega^{-\lambda_m - 1/a} \cdot c/(t\omega), & t\omega \geqslant 1, \\ 0 & t\omega < 1. \end{cases}$$

Here we choose so that

$$c^{-1} = \frac{1}{t}\int_{t\omega \geqslant 1} \omega^{-(1+1/a)}\, ds \sim 1.$$

Then $\tilde{\Gamma}_m(u_m)=1$ and

$$J(t, u_m) \leqslant C\left(\int_{t\omega \geqslant 1} \omega^{-(\lambda+1/a)p}\, ds\right)^{1/p} \leqslant Ct^{\theta_m}.$$

It remains to consider the determinant $\det[\tilde{\Gamma}_r(u_k)]$. Clearly

$$\tilde{\Gamma}_r(u_k)=\frac{c}{t}\int_{t\omega \geqslant 1} \omega^{\lambda_r-\lambda_k-1}\frac{ds}{\omega^{1/a}}=\frac{2c}{a}t^{\theta_k-\theta_r}\frac{1}{1-\Delta_{r,k}}$$

where

$$\Delta_{r,k}=\theta_r-\theta_k.$$

Note that $\Delta_{r,k}<1$ since $\lambda_r<a-1/q$ and $\lambda_k>-1/q$. Using the element on place (M, M) as pivot element we see that

$$\begin{aligned}
D_M &= \det\left[\frac{1}{1-\Delta_{r,k}}\right]_{r,k=1,\dots,M}\\
&= \det\left[\frac{1}{1-\Delta_{r,k}}-\frac{1}{(1-\Delta_{r,M})(1-\Delta_{M,k})}\right]\\
&= \det\left[\frac{\Delta_{r,m}\Delta_{M,k}}{(1-\Delta_{r,k})(1-\Delta_{r,M})(1-\Delta_{M,k})}\right]_{r,k=1,\dots,M-1}\\
&= \prod_{r=1}^{M-1}\frac{\Delta_{r,M}}{1-\Delta_{r,M}}\prod_{k=1}^{M-1}\frac{\Delta_{M,k}}{1-\Delta_{M,k}}\det\left[\frac{1}{1-\Delta_{r,k}}\right]_{r,k=1,\dots,M-1}\\
&= (-1)^{M-1}\prod_{k=1}^{M-1}\frac{\Delta_{M,k}^2}{1-\Delta_{M,k}^2}\cdot D_{M-1}.
\end{aligned}$$

Since $D_1=1$ we conclude that

$$D_M=(-1)^M\prod_{1\leqslant k<r\leqslant M}\frac{\Delta_{r,k}^2}{1-\Delta_{r,k}^2}.$$

Thus $D_M\neq 0$ and hence $\det[\tilde{\Gamma}_r(u_k)]\neq 0$ if all the numbers $\lambda_1, \dots, \lambda_M$ are different. This completes the proof.

LEMMA 10. *Let $\varphi_1, \dots, \varphi_M$ be locally integrable functions such that*

$$\lim_{|s|\to\infty} |s|^{-a\lambda_m}\varphi_m(s)$$

exist and are non-vanishing for all $m=1, \dots, M$. Put $\theta_m=\lambda_m+1/aq$ and assume that all λ_m are different and that $0<\theta_m<1$. Then the functionals Γ_m defined by φ_m are strongly independent.

Proof. Let $\tilde{\Gamma}_m$ be defined by the previous lemma and put

$$L = \lim_{|s| \to \infty} |s|^{-a\lambda_m} \phi_m(s)$$

Moreover we write $\Delta = L\tilde{\Gamma}_m - \Gamma_m$. Then the result follows if we can prove that $\tilde{\Gamma}_m$ dominates Δ. Thus let ε be a given positive number. Then we can find R so that $|L - \phi_m/\psi_m| \leqslant \varepsilon$ for all s with $|s| \geqslant R$. Then it follows that

$$N(t, \Delta) \leqslant C\left(\varepsilon^q \int_{|s| \geqslant R} \left(\frac{\psi_m}{\omega_t}\right)^q ds + \int_{|s| < R} |L\psi_m - \phi_m|^q \, ds\right)^{1/q}$$

Thus we conclude that

$$\limsup_{t \to 0} t^{\theta_m} N(t, \Delta) \leqslant C\varepsilon$$

for all $\varepsilon > 0$. Thus $\tilde{\Gamma}_m$ dominates Δ. The result now follows.

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